Math 210C Lecture 10 Notes

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1 Discrete Valuations and Completion

1.1 Equivalent condition for DVRs

Let's finish the proof of the following theorem.

Theorem 1.1. *If A is a noetherian domain, then A is Dedekind if and only if $A_{\mathfrak{p}}$ is a DVR for all nonzero prime ideals $\mathfrak{p}$ of A.*

Proof. ($\Longleftarrow$): Last time, we defined $A' = \bigcap_{\mathfrak{p} \neq 0} A_{\mathfrak{p}} \subseteq Q(A)$ and showed that $A' = A$.

We now show that A has Krull dimension ≤ 1. Let $\mathfrak{q} \subseteq A$ be a nonzero prime. Then $\mathfrak{q} \subseteq \mathfrak{m}$ for some maximal ideal $|mfm$. Now $0 \neq \mathfrak{q}A_{\mathfrak{m}}$, which equals $\mathfrak{m}A_{\mathfrak{m}}$ since $A_{\mathfrak{m}}$ is a DVR. Also, $\mathfrak{q} = \mathfrak{q}A_{\mathfrak{m}} \cap A = \mathfrak{m}A_{\mathfrak{m}} \cap A = \mathfrak{m}$. So A indeed has Krull dimension 1.

It remains to show that A is integrally closed. Let $a \in Q(A)$ be integral over A. THen a is integral over $A_{\mathfrak{p}}$ for all $\mathfrak{p} \neq 0$. Each $A_{\mathfrak{p}}$ is integrally closed (as DVRs are Dedekind domains), so $a \in A_{\mathfrak{p}}$ for all $\mathfrak{p}$. So $a \in A' = A$. □

Example 1.1. Let K be a number field. Then the ring of integers O_K is a Dedekind domain. If $\mathfrak{p} \subseteq O_K$ is prime, then $(O_K)_{\mathfrak{p}}$ is a DVR. Llet $\hat{O}_{K,\mathfrak{p}} = \varprojlim_n O_K/\mathfrak{p}^n$. This is local with maximal ideal $\mathfrak{p}\hat{O}_{K,\mathfrak{p}}$. This is called the **completion** of O_K with respect to $\mathfrak{p}$.

1.2 Discrete valuations

Definition 1.1. Let K be a field. A **discrete valuation** on K is a surjective function $v : K \to \mathbb{Z} \cup \{\infty\}$ such that

1. $v(a) = \infty \iff a = 0$,
2. $v(ab) = v(a) + v(b)$,
3. $v(a + b) \geq \min(v(a), v(b))$ for all $a, b \in K$.

The quantity $v(a)$ is called the **valuation** of a.

Example 1.2. Let p be a prime number. The **p-adic valuation** is $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ is given as $v_p(p^k \frac{a}{b}) = k$, where $k \in \mathbb{Z}$ and $a, b \in \mathbb{Z}$ with $p \nmid ab$. Check the definition:

1. $v_p(0) = \infty$.
2. $v_p(pQ^k \frac{a}{b} p^\ell \frac{c}{d}) = v_p(p^{k+\ell} \frac{ac}{bd}) = k + \ell$
3. $$\begin{aligned} v_p(p^k \frac{a}{b} + p^\ell \frac{c}{d}) &= v_p \left(p^{\min(k,\ell)} \left(p^{k-\min(k,\ell)} \frac{a}{b} + p^{\ell - \min(k,\ell)} \frac{c}{d} \right) \right) \\ &\geq \min(k, \ell) \\ &= \min \left(v_p \left(p^k \frac{a}{b} \right), v_p \left(p^\ell \frac{c}{d} \right) \right) \\ &= \min(k, \ell). \end{aligned}$$

Example 1.3. Let F be a field and consider $F(t)$. Define $v_\infty : F(t) \to \mathbb{Z} \cup \{\infty\}$ sending $v_\infty(f/g) = \deg g - \deg f$ for $g \neq 0$. Also define $\deg(0) = -\infty$. If f is irreducible in $F[t]$, then set $v_f(f^k \frac{g}{h}) = k$, where $f \nmid gh$. In $F[t^{-1}]$, $v_{t^{-1}} = v_\infty$.

Lemma 1.1. *Let A be a Dedekind domain, and let $\mathfrak{p}$ be a nonzero prime. Define the $\mathfrak{p}$-adic valuation $v_p : Q(A) \to \mathbb{Z} \cup \{\infty\}$: for $a \in Q(A)$, $aA_\mathfrak{p} = \mathfrak{p}^{v_\mathfrak{p}(a)} A_\mathfrak{p}$. Then $v_\mathfrak{p}$ is a discrete valuation on $Q(A)$.*

Lemma 1.2. *Let v be a discrete valuation. Then*

1. $v(1) = 0$, *$v(-a) = v(a)$,*
2. $v(a + b) = \min(v(a), v(b))$ *if* $v(a) \neq v(b)$.

Definition 1.2. Let v be a discrete valuation on a field K. The **valuation ring** of v is $O_v = \{a \in K : v(a) \geq 0\}$.

Proposition 1.1. *O_v is a DVR with maximal ideal $\mathfrak{m}_v = \{a \in K : v(a) \geq 1\}$.*

Proof. As an exercise, check that O_v is a ring.

$\mathfrak{m}_v$ is an ideal: if $a, b \in \mathfrak{m}_v$, $v(a + b) \geq \min(v(a), b(b)) \geq 1$. If $a \in O_v$ and $b \in \mathfrak{m}_v$, then $v(ab) = v(a) + v(b) \geq 0 + 1 = 1$.

O_v is local with maximal ideal $\mathfrak{m}_v$: If $a \in O_v \setminus \mathfrak{m}_v$, then $v(a^{-1}) = -v(a) = 0$, so $a^{-1} \in O_v$. So $a \in O_v^\times$.

O_v is a DVR: Let $a \in \mathfrak{m}_v$, and let $v(a) = n \geq 1$. Let $\pi \in \mathfrak{m}_v$ with $v(\pi) = 1$. Then $v(a\pi^{-n}) = 0$, so $a\pi^{-n} = u \in O_v^\times$. So $a = u\pi^n$, which gives $(a) = (\pi^n) = \mathfrak{m}^n$. Every ideal is generated by a power of π, so O_v is a DVR. □

Example 1.4. The p-adic integers $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ has a unique maximal ideal generated by $p = (\ldots, p, p, p)$. The p-adic valuation on $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ is $v_p((a_n)_{n \geq 1})$ is the largest k such that $a_k = 0$. The valuation on $\mathbb{Q}_p$ is the defined as $v_P(x/y) = v_p(x) - v_p(y)$, where $x, y \in \mathbb{Z}_p$.

If we set $|x|_p = p^{-v_p(x)}$, then $d(x, y) = |x - y|$ defines a metric on $\mathbb{Q}_p$. This matches up with the profinite topology on $\mathbb{Z}_p$.

1.3 Completion

Definition 1.3. Let (I_n) be a descending sequence of ideals of a ring R such that $\bigcap_{n=1}^{\infty} I_n = (0)$, the **completion** is $\hat{R} = \varprojlim_n R/I_n$. If $I_n = I^n$, where I is an idael, then $\hat{R}_I = \varprojlim_n R/I^n$ is called the ***I*-adic completion**.

Example 1.5. If R is commutative and $I = \mathfrak{p}$ with $\mathfrak{p}$ prime, then $\hat{R}_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}\hat{R}_{\mathfrak{p}}$.

Let A be a Dedekind domain with $K = Q(A)$. Let L/K be a finite, separable extension, and let B be the integral closure of A in L. Then if $\mathfrak{p} \subseteq A$, then $\mathfrak{p}B \subseteq B$ factors as $\mathfrak{p}B = P_1^{e_1} \cdots P_g^{e_g}$, where $e_i \geq 1$, $g_i \geq 1$, and P_i are distinct primes.

Definition 1.4. If $\mathfrak{p} \subseteq A$ is prime and $P \subseteq B$ lies over it, then $P/\mathfrak{p}$ is **ramified** if $\mathfrak{p}B_{\mathfrak{p}} = P^e B_{\mathfrak{p}}$ with $e > 1$. Otherwise, it is **unframified**. The field extension $B/\mathfrak{p}$ $A/\mathfrak{p}$ has the **residue degree** $f_{P/\mathfrak{p}} = [B/P : A/\mathfrak{p}]$.

Theorem 1.2. $[L : K] = \sum_{i=1}^{g} e_{P_i/\mathfrak{p}} f_{P_i/\mathfrak{p}}$.